Derivations with the identity sign

We need some new rules governing our identity predicate, "=". Now, our ordinary predicate logic rules apply to formulas with the identity sign; for example, the following is an acceptable use of universal out:

$\forall x(Fx \rightarrow x{=}a)$

$Fb \rightarrow b{=}a$

But we need some distinctive rules for "=", since "=" is a special predicate.

First, we need a rule that allows us to *use* an identity sentence. If we have

a=b

and

Fa

available, we should be able to infer:

Fb

After all, the predicate "F" applies to a, and a is the same thing as b. Similarly, if we had "a=b", and "Fb", we should be able to infer "Fa". Likewise, the following should be an acceptable inference:

a=b
Rac

Rbc

In general, we should be able to "substitute for identicals". That is, if we have "a=b" available, then in any formula, we should be able to change "a"s to "b"s, since a and b are the same thing; and we should also be able to change "b"s to "a"s. This rule we call "Leibniz's Law", and summarize as follows:

LL

n=m	n=m	* n and m are any two names
ϕ[n]	ϕ[m]	* ϕ[n] is any formula containing some "n"s; ϕ[m] is any formula containing some "m"s
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ϕ[m/n]	ϕ[n/m]	* ϕ[m/n] is ϕ[n] with some or all of the "n"s changed to "m"s; ϕ[n/m] is ϕ[m] with some or all of the "m"s changed

to “n”s

Notice that we are allowed to leave some of the “n”s when we change “n”s to “m”s. For example, suppose “H” stands for “hates”, “s” stands for “Superman”, and “c” stands for “Clark”. Then the following are all correct applications of Leibniz’s Law:

s=c	s=c	s=c
Hss	Hss	Hss
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Hsc	Hcs	Hcc

In the first two cases, we left in some “s”s; in the last one, we took out both “s”s. And these correspond to intuitively valid arguments: if Superman and Clark Kent are one and the same person, and if Superman hates Superman, then it should follow that Superman hates Clark, Clark hates Superman, and Clark hates Clark.

LL applies to complex formulas as well as atomics. For example, the following would be a legitimate use of LL:

1. ∀x(Fx→Ga)
2. a=b
3. ∀x(Fx→Gb) 1,2 LL

LL allows us to *use* identities (formulas like “a=b”) if we already have them in our derivations. We still need a way to get identities into our derivations if we have none to begin with. And there is just one way to do this. Even if we know nothing about a certain thing, a, we should be able to infer that a=a. In fact, this inference needs no premises whatsoever, since *every* object is identical to itself. So our second rule, the rule of “Reflexivity”, is as follows:

Rx

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n=n * n can be any name

Notice that the only annotation needed for a use of this rule is “Rx”. No line numbers are needed, because Rx needs no “input”.

Here’s an example of a derivation using our two rules:

1. ∃x(x=x) → ∃xRax Pr.

2.	a=b	Pr.
3.	~~Show~~ $\exists x Rbx$	DD
4.	a=a	Rx
5.	$\exists x(x=x)$	4, $\exists$I
6.	$\exists x Rax$	1,5 $\rightarrow$ O
7.	Rac	6, $\exists$O
8.	Rbc	2,7 LL
9.	$\exists x Rbx$	8, $\exists$I

I now want to introduce two derived rules. The first expresses the "transitivity of identity":

Trans

n=m   * n, m, and l are any
m=l   * three names
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n=l

It is a "derived" rule because we can actually get to its conclusion using the two rules we already have. For example, suppose we have "a=b" and "b=c" as premises. We can then derive "a=c" as follows.

1.	a=b	
2.	b=c	
3.	~~show~~ a=c	DD
4.	a=c	LL 1,2

The other derived rule I want to introduce is "symmetry":

Sym

n=m
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m=n

It too is a derived rule, in virtue of derivations like the following:

1.	a=b	Pr.
2.	~~Show~~ b=a	DD
3.	a=a	1,1 LL
4.	b=a	1,3 LL

Here, in the first use of LL, we used line 1 twice. We let "n" be "a" and we let "m" be "b"; and we let our $\phi[n]$ be line 1 itself (i.e., "a=b"). LL says that we can change "b"s to "a"s in "a=b"; so we get "a=a". Then in moving from 1 and 3 to 4, we changed one of the "a"s in line 3 to a "b". (Recall that we don't need to change all of the "a"s to "b"s.)

Let's now consider some examples:

1.	~~Show~~ $\forall x \forall y[x=y \rightarrow \forall z(Rxz \leftrightarrow Ryz)]$	UD
2.	~~Show~~ $\forall y[a=y \rightarrow \forall z(Raz \leftrightarrow Ryz)]$	UD
3.	~~Show~~ $a=b \rightarrow \forall z(Raz \leftrightarrow Rbz)$	CD
4.	$a=b$	As.
5.	~~Show~~ $\forall z(Raz \leftrightarrow Rbz)$	UD
6.	~~Show~~ $Rac \leftrightarrow Rbc$	DD
7.	~~Show~~ $Rac \rightarrow Rbc$	CD
8.	Rac	As
9.	~~Show~~ Rbc	DD
10.	Rbc	4,8, LL
11.	~~Show~~ $Rbc \rightarrow Rac$	CD
12.	Rbc	As.
13.	~~Show~~ Rac	DD
14.	Rac	12,4 LL
15.	$Rac \leftrightarrow Rbc$	7,11 $\leftrightarrow$I

1.	~~Show~~ $\forall x[Fx \leftrightarrow \forall y(y=x \rightarrow Fy)]$	UD
2.	~~Show~~ $Fa \leftrightarrow \forall y(y=a \rightarrow Fy)$	DD
3.	~~Show~~ $Fa \rightarrow \forall y(y=a \rightarrow Fy)$	CD
4.	Fa	As.
5.	~~Show~~ $\forall y(y=a \rightarrow Fy)$	UD
6.	~~Show~~ $b=a \rightarrow Fb$	CD
7.	$b=a$	As.
8.	~~Show~~ Fb	DD
9.	Fb	7,4, LL
10.	~~Show~~ $\forall y(y=a \rightarrow Fy) \rightarrow Fa$	CD
11.	$\forall y(y=a \rightarrow Fy)$	As.
12.	~~Show~~ Fa	DD
13.	$a=a$	Rx.
14.	$a=a \rightarrow Fa$	11,$\forall$O
15.	Fa	14,13,$\rightarrow$O
16.	$Fa \leftrightarrow \forall y(y=a \rightarrow Fy)$	3,10 $\leftrightarrow$I

1.	$\exists x \forall y(Fy \rightarrow y=x)$	Pr.
2.	$\exists x(Fx \& Gx)$	Pr.
3.	~~Show~~ $\forall x(Fx \rightarrow Gx)$	UD
4.	~~Show~~ $Fa \rightarrow Ga$	CD

5.	Fa	As.
6.	~~Show~~ Ga	DD
7.	Fb&Gb	2,∃O
8.	∀y(Fy→y=c)	1,∃O
9.	Fb	7,&O
10.	Fb→b=c	8,∀O
11.	b=c	10,9 →O
12.	Fa→a=c	8,∀O
13.	a=c	12,5 →O
14.	c=a	13,Sym
15.	b=a	11,14 Trans
16.	Gb	7,&O
17.	Ga	15,16, LL

1.	~∃x∃y ~x=y	Pr.
2.	~~Show~~ ∃xFx↔∀xFx	DD
3.	~~Show~~ ∃xFx→∀xFx	CD
4.	∃xFx	As.
5.	~~Show~~ ∀xFx	UD
6.	~~Show~~ Fa	ID
7.	~Fa	As
8.	~~Show~~ x	DD
9.	Fb	4,∃O
10.	∀x~∃y~x=y	1,~∃O
11.	~∃y~a=y	10, ∀O
12.	∀y~~a=y	11, ~∃O
13.	~~a=b	12, ∀O
14.	a=b	13, DN
15.	~Fb	14,7 LL
16.	x	9,15 xI
17.	~~Show~~ ∀xFx→∃xFx	CD
18.	∀xFx	As.
19.	~~Show~~ ∃xFx	DD
20.	Fa	18,∀O
21.	∃xFx	20, ∃I
22.	∃xFx↔∀xFx	3,17 ↔I