Translations with the identity sign

We already know how to symbolize:

> Only teachers can change grades.

If we let ‘C’ stand for ‘can change grades’, then we have either:

(1) $\sim\exists x(\sim Tx \,\&\, Cx)$

or

(2) $\forall x(Cx \rightarrow Tx)$

But there are similar sentences that we cannot yet satisfactorily translate using our present system of logic. The sentence:

> Only Ted can change grades.

means “no one besides Ted can change grades” -- that is, “For any person x, if x can change grades, then x must be Ted”. So we want to symbolize it, by analogy to (2), as something like:

> $\forall x(Cx \rightarrow x \text{ is Ted})$

But how do we symbolize “x is (identical to) Ted”? We don’t want to say “Tx”, because ‘Ted’ is a name, and so is symbolized by a lowercase letter, not an uppercase ‘T’.

Our solution is to introduce a special two-place predicate to stand for ‘is identical to’. We *could* just pick ‘I’; the result would then be:

> $\forall x(Cx \rightarrow Ixt)$

But since the identity predicate has certain special features, we will choose a new symbol: “=”. And in an atomic sentence where this new predicate letter occurs, we will put the terms on either side of the “=” sign, rather than putting both after it. Thus, the symbolization of our original sentence is:

> $\forall x(Cx \rightarrow x=t)$

This is to be read: “for any x, if x can change grades, then x is identical to Ted”.

Another example:

> The only truly great player who plays in the NBA is Michael Jordan.

Recall that

The only dangerous rats are rabid rats

is symbolized

$\forall x[(Dx \,\&\, Rx) \rightarrow (Bx \,\&\, Rx)]$

(‘B’ stands for ‘is rabid’.) By analogy, if we let ‘S’ stand for ‘is a superstar’, ‘P’ stand for ‘plays in’, ‘n’ stand for ‘the NBA’, and ‘m’ stand for ‘Michael Jordan’, then we have:

$\forall x[(Sx \,\&\, Pxn) \rightarrow x{=}m]$

Another example. Recall that in some cases, ‘and’ doesn’t mean conjunction, but rather disjunction. For example,

Only cats and dogs are good pets

does *not* mean

$\forall x[Gx \rightarrow (Cx\&Dx)]$

(which says that every good pet is both a cat and a dog), but rather:

$\forall x[Gx \rightarrow (Cx \vee Dx)]$

Similarly, we symbolize:

Only Sprittle, Chim Chim, and Trixie stow away in the Mock Five

as:

$\forall x[\ Sxm \rightarrow (x{=}s \vee x{=}c \vee x{=}t)]$

(That is: for all x, if x stows away in the Mock Five, then x is either Sprittle, Chim Chim, and Trixie.)

Note that it would be wrong to symbolize this sentence as:

$\forall x[\ Sxm \rightarrow (x = s \vee c \vee t)$

for this doesn’t make any sense. The string of symbols “s∨c∨t” isn’t grammatical, since the ∨ can only go between *formulas*, and ‘s’, ‘c’, and ‘t’ are *terms*.

Next, let’s think about:

Every lawyer hates every other lawyer.

If we just wrote:

$\forall x(Lx \rightarrow \forall y[(Ly \rightarrow Hxy]$

this would be incorrect, because this says that every lawyer hates every lawyer. To say that every lawyer hates every *other* lawyer, we write:

$\forall x(Lx \rightarrow \forall y[(Ly \ \& \ {\sim}x=y) \rightarrow Hxy]$

Now try:

Everyone who loves someone else loves everyone.

The new feature of this sentence is the occurrence of 'someone *else*'. To say that x loves someone else is to say that x loves someone other than herself -- that is, that x loves someone other than x -- that is, that x loves someone *not* identical to x. Thus, we have:

$\forall x[\exists y({\sim}y=x\&Lxy) \rightarrow \forall yLxy]$

(That is: "for all x, if x loves someone else (that is, if there is a y that is not identical to x that x loves), then x loves everyone).

It will be convenient to abbreviate "~x=y" as $x \neq y$.

Let's put some of these ideas together in a fairly tricky problem:

If a person shares a solitary confinement cell with a guard, then they are the only people in the cell.

Notice the 'a's -- these are universal quantifiers, since the sentence is saying "*any* person that shares *any* cell with *any* guard is such and such". Let's let 'C' stand for 'is a solitary confinement cell', let 'Sxyz' stand for 'x shares y with z' (it's a three place predicate), and let 'I' stand for 'is in'. We then have:

$\forall x\forall y\forall z[(Px \ \& \ Cy \ \& \ Gz \ \& \ Sxyz) \rightarrow \forall x_1([Px_1 \ \& \ Ix_1y] \rightarrow [x_1=x \vee x_1=z])]$

Read: "For any x, y, and z, if x is a person, y is a cell, Z is a guard, and x shares y with z (i.e. if the person shares the cell with the guard), then, take any thing, x_1; if this new x_1 is also a person that is in y (i.e., in the cell mentioned earlier), then x_1 must be either x or z (i.e., this new person must be either the original person or the guard).

Let's now think about the sentence:

There are at least two dinosaurs.

At first, we might try:

$\exists x \exists y(Dx \ \& \ Dy)$

But this would be true even if there were only one dinosaur, since x and y could be assigned the same dinosaur. The identity sign to the rescue:

$\exists x \exists y(Dx \ \& \ Dy \ \& \ x \neq y)$

This says that there are two *different* objects, x and y, each of which are dinosaurs. If we want to say that there are at least *three* dinosaurs, it is a little trickier:

$\exists x \exists y \exists z(Dx \ \& \ Dy \ \& \ Dz \ \& \ x \neq y \ \& \ x \neq z \ \& \ y \neq z)$

since we need to go through x, y, and z, and say of every pair of two that they are not identical. I think you can see that to say there are at least *four* dinosaurs would take even longer.

Still trickier is saying:

There are *exactly* two dinosaurs.

(That is, that there are two dinosaurs, but no more than two.) The way to do it is to say that there are at least two dinosaurs, which are such that every dinosaur is one of those two:

$\exists x \exists y(Dx \ \& \ Dy \ \& \ x \neq y \ \& \ \forall z[Dz \rightarrow (z=x \vee z=y)])$

What about “there are no more than two dinosaurs”? First symbolize “there are at least three dinosaurs”, and then negate the answer.

What about “there are between two and six dinosaurs”? First symbolize “there are at least two dinosaurs”. Then symbolize “there are no more than six dinosaurs”. Conjoin the two results (i.e. put a “&” between them), and that’s the answer.

Let’s move to the final, and most powerful thing we can do with the = sign. Let’s start with:

Some 8-foot tall man is happy.

which, as we know, is symbolized:

$\exists x(Ex \ \& \ Mx \ \& \ Hx)$

(where ‘E’ stands for ‘is eight feet tall’).

But what about:

The 8-foot tall man is happy.

? Before, we would just use ‘e’ to symbolize ‘the 8-foot tall man’, but now we can break this up. We want to consider the one and only 8-foot tall man. What we do is begin as we did for “Some 8-foot tall man is happy” but say something more about that 8 foot tall man -- namely, that he’s the only 8-foot tall man. So, in all, we say that there is someone that is an 8-foot tall man, is the *only* 8-foot tall man, and is happy:

$$\exists x[Ex \& Mx \& \forall y([Ey \& My] \rightarrow x=y) \& Hx]$$

In general, we have the following rule:

“the F is G” is symbolized: “$\exists x[Fx \& \forall y(Fy \rightarrow x=y) \& Gx]$

Let’s call terms like “the F” *definite descriptions*. This method of understanding the meaning of definite descriptions was invented by Bertrand Russell.

We sometimes want to symbolize sentences with two or more definite descriptions, such as:

The 8-foot tall man drove the 20-foot long limo.

Here’s how: (letting ‘T’ stand for ‘is twenty feet long’)

$$\exists x\exists y[Ex \& Mx \& \forall z([Ez \& Mz] \rightarrow x=z) \& Ty \& Ly \& \forall z([Tz \& Lz] \rightarrow y=z) \& Dxy]$$

To avoid getting lost, remember that x is the man, and y is the limo. The first $\forall z$ part says that x is the *only* eight-foot tall man; the second $\forall z$ part says that y is the *only* 20-foot limo, and the final Dxy says that x (the man) drove y (the limo).

An interesting problem arises with negations of sentences involving definite descriptions:

The president is not bald.

Does this mean:

The president is such that he’s non-bald.

which is symbolized as follows:

$$\exists x[Px \& \forall y(Py\rightarrow x=y) \& \sim Bx]$$

? Or does it mean:

It is not the case that the President is bald

which is symbolized thus:

$$\sim\exists x[Px \& \forall y(Py\rightarrow x=y) \& Bx]$$

? The answer is that the original sentence is simply ambiguous. Symbolizing it the first way is called “giving the description wide scope (relative to the ~)”, since the ~ is inside the scope of the ∃x. Symbolizing it in the second way is called “giving the description *narrow* scope (relative to the ~)”, because the ∃x is inside the scope of the ~.

What is the difference in meaning between these two symbolizations? The first says that there really is a unique president, and adds that he is not bald. So the first implies that there’s a unique president. The second merely denies that there is a unique president, who is bald. That doesn’t imply that there’s a unique president. It would be true if there’s a unique president who is not bald, but it would also be true in two other cases:

i) there’s no president
ii) there is more than one president

A similar ambiguity arises with the following sentence:

The round square does not exist.

We might think to symbolize it:

$$\exists x[Rx \ \&\ Sx \ \&\ \forall y([Ry \ \&\ Sy] \rightarrow x=y) \ \&\ {\sim}Ex]$$

letting “E” stands for “exists”. In other words, we might give the description wide scope. But this is wrong, because it says *there is* a certain round square that doesn’t exist, and that’s a contradiction. This way of symbolizing the sentence corresponds to reading the sentence as saying:

The thing that is a round square is such that it does not exist

But that isn’t the most natural way to read the sentence; rather, the sentence would usually be interpreted to mean:

It is not true that the round square exists.

-- that is, as the negation of “the round square exists”:

$${\sim}\, \exists x[Rx \ \&\ Sx \ \&\ \forall y([Ry \ \&\ Sy] \rightarrow x=y) \ \&\ Ex]$$

with the ~ out in front. Here we’ve given the description narrow scope. Notice also that saying that x exists at the end is redundant, so we could simplify to:

$${\sim}\, \exists x[Rx \ \&\ Sx \ \&\ \forall y([Ry \ \&\ Sy] \rightarrow x=y)]$$

Again, notice the moral of these last two examples: if a definite description occurs in a sentence with a ‘not’, the sentence may be ambiguous: does the ‘not’ apply to the entire rest of the sentence, or merely to the predicate?

Let’s try two more examples with definite descriptions. The first shows that the term ‘the’ needn’t always be present in a definite description:

Ted’s father lives in Philadelphia.

This sentence clearly means:

The father of Ted lives in Philadelphia.

Thus, using ‘F’ to symbolize ‘is a father of’, we have:

$$\exists x[Fxt \mathbin{\&} \forall y(Fyt \rightarrow x=y) \mathbin{\&} Lxp]$$

Finally, we have a definite description occurring within a more complicated sentence:

The only person that is proud of Ted is his father.

The definite description is ‘his father’, which means again ‘the father of Ted’. So we have (with ‘P’ standing for ‘is a person’ and ‘R’ standing for ‘is proud of’):

$$\exists x[Fxt \mathbin{\&} \forall y(Fyt \rightarrow x=y) \mathbin{\&} \forall y([Py \mathbin{\&} Ryt] \rightarrow x=y)]$$

This says: “There is an x, that is the one and only father of Ted; and moreover, any person that is proud of Ted is x.”