Measurement Theory

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Properties seminar

1. The problem of quantity

"Qualities" don't come in degrees, "quantities" do.

Problem 1 What are the fundamental facts of quantity like, which enable them to be spoken of and theorized about using numbers?

Simplest theory of quantity: quantities are relations to numbers. E.g., the fundamental property of mass is, perhaps, the mass-in-kilograms relation, which holds between concrete object x and real number r iff x's mass is r kg.

Objections: privileges a single unit of mass, involves real numbers in the facts of mass. (Why is the latter bad? "real numbers are abstract and therefore causally inert"; "real numbers don't fundamentally exist"; "real numbers are constructed entities, and constructed entities can't be involved (qua the construction) in fundamental facts".)

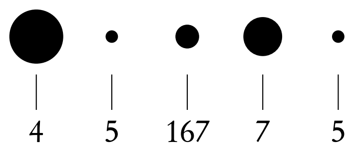
Problem 2 What is the deficiency of statements of quantity that "don't make sense" because of missing quantitative structure?

1. The mass of object o is 5
2. The mass of object o is 5 g
3. The mass of object o is greater than the mass of object p
4. The mass of object o is twice that of the mass of object p
5. The mass of object o is greater than the charge of object p
6. Smith is witty to degree 6.808942 in the Johnson scale
7. The wit of Smith is greater than that of Jones
8. The wit of Smith is twice the wit of Jones

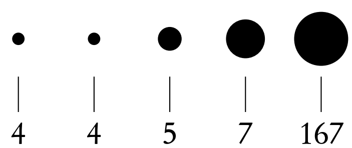
2. Using numbers to represent quantities

Basic idea of measurement theory: numbers can be used to *represent* a physical system when the numbers share the same *structure* as the physical system.

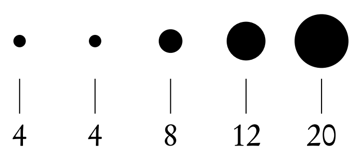
Example: assigning numbers to massive objects:



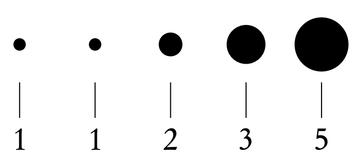
(1) x is assigned the same number as y iff x and y have the same mass



(2) x is assigned a greater number than y iff x is more massive than y



(3) The sum of the numbers assigned to x and y equals the number assigned to z iff x and y's combined masses equal z's mass



In sum: we can assign numbers to objects in a way that encodes information about the objects' nonnumeric properties. Different assignments can encode different amounts of information.

3. Relational structures, homomorphisms and representation theorems

Relational structure: an n-tuple $\langle A, R_1 \dots R_n \rangle$, where A is a set and $R_1 \dots R_n$ are relations on that set.

Homomorphism ("structure-preserving function"): a function f is a homomorphism from one relational structure $\langle A, R_1 \dots R_n \rangle$ into another $\langle B, S_1 \dots S_n \rangle$ iff f is a function from A into B such that for each R_i, $R_i(x_1 \dots x_m)$ iff $S_i(f(x_1) \dots f(x_m))$

- Think of the nonnumeric facts as a relational structure. E.g., the facts of mass are $\langle A, \succeq, C \rangle$, where A is the set of the five massive objects above, $\succeq$ is the two-place being at-least-as-massive-as relation, and C is the three-place combining-to-equal-in-mass relation.
- Think of the mathematical facts as another relational structure. E.g. $\langle \mathbb{R}, \geq, R_+ \rangle$, where $\mathbb{R}$ is the set of real numbers, $\geq$ is the greater-than-or-equal-to relation on those numbers, and R_+ is the addition relation on real numbers: $R_+ xyz$ holds iff $x + y = z$.
- A mathematical structure will be useful tool to represent a nonmathematical structure if there is a homomorphism from the nonmathematical structure into the mathematical structure
- Homomorphic structures have analogous structure. We can use a homomorphism to extract information about the nonmathematical structure from information about the mathematical structure. A particular homomorphism is just a scale.

Representation theorems tell us that homomorphisms exist from certain nonmathematical structures into certain mathematical structures

4. Uniqueness theorems

Uniqueness theorems tell us how unique those homomorphisms are

Scale type	Preserves	Transformations
Ratio	ratios	similarity ($f = kg$)
Interval	ratios between intervals	affine ($f = kg + a$)
Ordinal	order	monotone

5. Assumptions made

$$
\begin{aligned}
M^n xy =_{\mathrm{df}} \text{ for some } & y_1 \ldots y_n : \\
& y_1 = y, \\
& C(y, y_i, y_{i+1}) \text{ for } 1 \leq i < n, \text{ and} \\
& y_n = x
\end{aligned}
$$

Archimedean assumption: For any x and y, if $x \succeq y$ then for some positive integer n and some z, $M^n zy$ and $z \succeq x$

A typical set of assumptions for mass:

- $\succeq$ is transitive and strongly connected (i.e. $x \succeq y$ or $y \succeq x$ holds for each x and y)
- C is "commutative" and "associative" in that:

  if $C(x, y, a)$ then $C(y, x, a)$

  if $C(x, y, a)$ and $C(a, z, b)$ and $C(y, z, c)$ then $C(x, c, b)$
- Adding the same mass preserves $\succeq$, in that:

  if $x \succeq y$, and if $C(x, z, x')$ and $C(y, z, y')$, then $x' \succeq y'$
- if $C(x, y, z)$ then $z \succ x$ (mass is never negative)
- Archimedean assumption
- Existence of copies: for each x and integer n, there exists some y such that $M^n yx$

6. Sketch of proofs

$$
\begin{aligned}
& x \succ y =_{\mathrm{df}} y \not\succsim x \\
& M^n xy \text{ iff } g(x) = n\, g(y) \\
& x \succ y \text{ iff } g(x) > g(y) \qquad\qquad \text{(for any homomorphism } g\text{)}
\end{aligned}
$$

Representation theorem for mass, proof sketch: First half constructs a certain function f; the second half shows that f is a homomorphism. I'm only going to do the first half.

First arbitrarily pick some object $e \in A$ (the unit). Set $f(e) = 1$.

Now take any other $a \in A$. Suppose a happens to be exactly n times as massive as e, for some integer n (i.e. $M^n ae$). Then we must let $f(a) = n f(e) = n$.

Similarly, suppose e just happens to be n times as massive as a. Then we must let $f(a) = \frac{1}{n}$.

Suppose that some mass is a "multiple" of both a and e—for some $x \in A$, and some integers m and n, $M^m xe$ and $M^n xa$. Then we must set $f(a) = \frac{m}{n}$. (Because $n f(a) = f(x) = m f(e) = m$.)

Otherwise we must let $f(a)$ be the least upper bound of certain fractions, namely the fractions $\frac{m}{n}$ when m copies of e is smaller than n copies of a. (Archimedean assumption needed!)

Uniqueness theorem for mass, proof sketch: Show that any homomorphism g is a scalar multiple of the homomorphism f that we constructed earlier—i.e., that for some real number k (the scaling factor), for all $a \in A$, $g(a) = k f(a)$.

How to choose the constant k? Well, k needs to equal $\frac{g(a)}{f(a)}$ for all a if we're to succeed; but $f(e) = 1$; so k must be $g(e)$. So what we must show is that $g(a) = g(e) f(a)$, i.e., $\frac{g(a)}{g(e)} = f(a)$, for all a.

Suppose for reductio that $\frac{g(a)}{g(e)} \neq f(a)$. Then either $\frac{g(a)}{g(e)} < f(a)$ or $\frac{g(a)}{g(e)} > f(a)$. I'll show that the first leads to a contradiction, and then stop.

Choose integers m and n such that $\frac{g(a)}{g(e)} < \frac{m}{n} < f(a)$. Choose an $x \in A$ whose mass is m times that of e, and an object y whose mass is n times that of a. That is, $M^m xe$ and $M^n ya$. (Note the use of the existence of copies.) Given (*), $g(x) = mg(e)$, and $g(y) = ng(a)$. So $\frac{g(a)}{g(e)} = \frac{mg(y)}{ng(x)}$; and so, since $\frac{g(a)}{g(e)} < \frac{m}{n}$, we know that $\frac{g(y)}{g(x)} < 1$ and so $y \prec x$. But given (*), $f(x) = mf(e)$ and $f(y) = nf(a)$, and so:

$$\frac{\frac{m}{n}}{f(a)} = \frac{f(x)}{f(y)}$$

But the left hand side of this is less than 1 (since $\frac{m}{n} < f(a)$) whereas the right hand side is greater than 1 (since $y \prec x$).

7. Kinds of quantities

Any other quantity for which there are relations obeying the same assumptions as $\succeq$ and C will obey the same representation and uniqueness theorems. For quantities with different characteristic relations, or similar relations but obeying different assumptions, different representation and uniqueness theorems will provable.

8. Measurement theory: metaphysics and epistemology

The philosophers of science who developed measurement theory were largely concerned with epistemic questions like: we can't *observe* correlations between physical objects and real numbers, so how can the use of real numbers be justified in terms of things we *can* observe? As we saw, the metaphysical concerns about quantity are different; but they too can be addressed using measurement theory.

- Possible answer to problem 1: the fundamental relations for a quantity are those relations in the nonmathematical structures ($\succeq$ and C in the case of mass). Talk of numbers is useful (and justified) because of the homomorphisms.

- Possible answer to problem 2: there are insufficient fundamental relations to prove the uniqueness theorems, in cases of insufficient structure